HYPERBOLICITY OF GENERAL DEFORMATIONS

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Abstract. This is the content of the talk given at the conference "Effective Aspects of Complex Hyperbolic Varieties", Aver Wrac'h, France, September 10-14, ′07. We present two methods of constructing low degree Kobayashi hyperbolic hypersurfaces in $\mathbb{P}^n$:

- The projection method
- The deformation method

The talk is based on joint works of the speaker with B. Shiffman and C. Ciliberto.

1. DIGEST on KOBAYASHI THEORY

1.1. Kobayashi hyperbolicity.

DEFINITION The *Kobayashi pseudometric* k_X on a complex space X satisfies the following axioms :

(i) On the unit disc Δ, the Kobayashi pseudometric k_Δ coincides with the Poincaré metric;

(ii) every holomorphic map $\varphi : \Delta \to X$ is a contraction: $\varphi^*(k_X) \le k_\Delta$;

(iii) k_X is the maximal pseudometric on X satisfying (i) and (ii).

REMARK Every holomorphic map $\varphi : X \to Y$ is a contraction: $\varphi^*(k_Y) \le k_X$.

DEFINITION X is called *Kobayashi hyperbolic* if k_X is non-degenerate :

$$k_X(p, q) = 0 \Longleftrightarrow p = q.$$

EXAMPLES $k_{\mathbb{C}^n} \equiv 0, \quad k_{\mathbb{P}^n} \equiv 0, \quad k_{\mathbb{T}^n} \equiv 0$, where $\mathbb{T}^n = \mathbb{C}^n/\Lambda$ is a complex torus,

whereas $\mathbb{C} \setminus \{0, 1\}$ is hyperbolic (the Schottky-Landau Theorem.)

1.2. Classical theorems.

According to the above definition and to Royden's Theorem, X is hyperbolic iff natural analogs of the classical Schottky and Landau Theorems hold for X.

Brody-Kiernan-Kobayashi-Kwack THEOREM
For a compact complex space X the following conditions are equivalent :

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- *X is Kobayashi hyperbolic;*
- *Little Picard Theorem holds for X :*

$$\forall f : \mathbb{C} \to X, \quad f = \text{const};$$

- *Big Picard Theorem holds for X :*

$$\forall f : \Delta \setminus \{0\} \to X \ \ \exists \bar{f} : \Delta \to X, : \bar{f}|(\Delta \setminus \{0\}) = f;$$

- *Montel Theorem holds for X : the space $HOL(\Delta, X)$ is compact.*

REMARK If X is hyperbolic then $\forall Y$, the space $HOL(Y, X)$ is compact.

DEFINITION Let M be a hermitian compact complex manifold. An entire curve $\varphi : \mathbb{C} \to M$ is called a *Brody curve* if

$$||\varphi'(z)|| \leq 1 = ||\varphi'(0)|| \quad \forall z \in \mathbb{C}\,.$$

Brody's THEOREM *M as above is hyperbolic iff it does not possess any Brody entire curve.*

Brody's STABILITY THEOREM
Every compact hyperbolic subspace X of a complex space Z admits a hyperbolic neighborhood. Consequently, every compact subspace $X' \subseteq Z$ sufficiently close to X is hyperbolic. In particular, if $X \subseteq \mathbb{P}^n$ is a hyperbolic hypersurface then every hypersurface $X' \subseteq \mathbb{P}^n$ sufficiently close to X is hyperbolic too.

1.3. Hyperbolicity of hypersurfaces in $\mathbb{P}^n$.

Kobayashi Problem ($'70$)

Is it true that a (very) general hypersurface X of degree $d \geq 2n-1$ in $\mathbb{P}^n$ is Kobayashi hyperbolic? In particular, is this true for a (very) general surface X in $\mathbb{P}^3$ of degree $d \geq 5$?

Hyperbolic surfaces in $\mathbb{P}^3$

THEOREM (McQuillen [9], Demailly-El Goul [3])
A very general surface X in $\mathbb{P}^3$ of degree $d \geq 21$ is Kobayashi hyperbolic.

For some recent advances in higher dimensions, see Y.-T. Siu [15] and E. Rousseau [10].

EXAMPLES
of small degree hyperbolic surfaces in $\mathbb{P}^3$

Concrete examples were found by
Brody-Green $'77$, $d = 2k \geq 50$,
Masuda-Noguchi $'96$, $d = 3e \geq 24$,

Khoai $'96$, $d \geq 22$,
Nadel $'89$, $d \geq 21$,
Shiffman-Z$'$ $'00$, $d \geq 16$,
El Goul $'96$, $d \geq 14$,
Siu-Yeung $'96$, **Demailly-El Goul** $'97$, $d \geq 11$,
J. Duval $'99$ [5], **Shirosaki-Fujimoto** $'00$ [6], $d = 2k \geq 8$:

$$Q(X_0, X_1, X_2)^2 - P(X_2, X_3) = 0\,, \tag{1}$$

where Q, P are generic homogeneous formes of degree k and $d = 2k$, respectively,
Shiffman-Z$'$ $'02$ [11], $d = 8$,
Shiffman-Z$'$ $'05$ [12], $d \geq 8$,
J. Duval $'04$ [4], $d = 6$.

Algebraic families of hyperbolic hypersurfaces $X_n \subseteq \mathbb{P}^n$ for any $n \geq 3$ were constructed e.g., by
Masuda-Noguchi $'96$,
Siu-Yeung $'97$,
Shiffman-Z$'$ $'02$ [13].

In these examples $\deg X_n$ grows quadratically with n, for instance, $\deg X_n = 4(n-1)^2$ [13]. Whereas the Kobayashi Conjecture suggests a linear growth of the minimal such degree. This leads to the following problem.

PROBLEM *Find a sequence of hyperbolic hypersurfaces $X_n \subseteq \mathbb{P}^n$ with $\deg X_n \leq Cn$ for some positive constant C.*

2. PROJECTION METHOD

2.1. Symmetric powers of curves as hyperbolic hypersurfaces.

PROPOSITION (Shiffman-Z$'$ $'00$ [14]**)** *The nth symmetric power $C^{(n)}$ of a generic smooth projective curve C of genus $g \geq 3$ is hyperbolic iff $g \geq 2n - 1$. In particular, the symmetric square $C^{(2)}$ is always hyperbolic.*

THEOREM (Shiffman-Z$'$ $'00$ [14]**)** *With C as before, let us consider an embedding $C^{(2)} \hookrightarrow \mathbb{P}^5$. Then a general projection S of $C^{(2)}$ to $\mathbb{P}^3$ is hyperbolic. The minimal degree of such a hyperbolic surface $S \subseteq \mathbb{P}^3$ is equal* 16.

EXAMPLE of degree 16: Let $C \subseteq \mathbb{P}^2 : x^4 - xz^3 - y^3z = 0$, and let $C^{(2)} \hookrightarrow \mathbb{P}^5$ be embedded via the natural embedding of the symmetric square of $\mathbb{P}^2$ in $\mathbb{P}^5$. Then a general projection of $C^{(2)}$ to $\mathbb{P}^3$ is a singular hyperbolic surface $S \subseteq \mathbb{P}^3$ of degree 16, with the double curve D of genus 142.

Let us explain in brief our methods. Let $V \hookrightarrow \mathbb{P}^5$ be a smooth hyperbolic surface, and let $\pi : V \to S \hookrightarrow \mathbb{P}^3$ be a projection. Then S has self-intersection along a double curve $D \subseteq S$. By the universal property of the normalization, there is a commutative diagram

$$\begin{tikzcd} V \arrow[rr, "\psi"] \arrow[dr, "\pi"'] & & S_{\text{norm}} \arrow[dl, "\nu"] \\ & S & \end{tikzcd}$$

where $\nu : S_{\text{norm}} \to S$ is the normalization. By Zariski's Main Theorem, $\psi : V \to S_{\text{norm}}$ is an isomorphism. Hence any entire curve $\varphi : \mathbb{C} \to S$ can be lifted to $V = S_{\text{norm}}$:

$$\begin{tikzcd} & V \arrow[d, "\nu"] \\ \mathbb{C} \arrow[ur, "\tilde{\varphi}"] \arrow[r, "\varphi"'] & S \end{tikzcd}$$

unless $\varphi(\mathbb{C}) \subseteq D$. Since V is hyperbolic, $\tilde{\varphi}$ =cst. Thus S is hyperbolic iff D is. A similar argument shows that S is always hyperbolic modulo D. In the proof of the above theorem we show that, for a general projection, D is hyperbolic indeed and so S is. Similarly, for the Cartesian square of a curve the following holds.

PROPOSITION (Shiffman-Z′ ′00 [14]) *Let C be a smooth projective curve of genus $g \geq 2$. Let us fix an embedding $V = C \times C \hookrightarrow \mathbb{P}^n$. Then the double curve $D \subseteq S$ of a general projection $V \to S \subseteq \mathbb{P}^3$ is irreducible of genus $g(D) \geq 225$, and S is a singular hyperbolic surface of degree ≥ 32.*

However for a non-generic projection, the double curve of the image surface can be neither irreducible nor hyperbolic.

EXAMPLE (Kaliman-Z′ ′01 [8]) Consider the smooth Fermat quartic

$$C : \, x^4 + y^4 + z^4 = 0 \quad \text{in} \quad \mathbb{P}^2 \,.$$

Then the product $V = C \times C$ admits a projective embedding and a projection to $\mathbb{P}^3$ such that the double curve D of the image surface $S \subseteq \mathbb{P}^3$ consists of 4 disjoint projective lines. Thus S is not hyperbolic whereas its normalization V is.

For 3-folds in $\mathbb{P}^4$ we have the following result.

THEOREM (Ciliberto-Z′ ′03 [2]) *For a general projective curve C of genus $g \geq 7$, we fix an embedding $C^{(3)} \hookrightarrow \mathbb{P}^7$. Then a general projection X of $C^{(3)}$ to $\mathbb{P}^4$ is a hyperbolic hypersurface in $\mathbb{P}^4$. This is also true for a general quintic $C \subseteq \mathbb{P}^2$ ($g = 6$) and a certain special embedding $C^{(3)} \hookrightarrow \mathbb{P}^7$ of degree 125. The latter is the minimal degree which can be achieved via the projection method using the symmetric cubes $C^{(3)}$.*

The proof goes as follows. It is shown that

• $C^{(3)}$ does not contain any curve of genus $< g$; in particular, it is hyperbolic.

• $X \subseteq \mathbb{P}^4$ is hyperbolic iff the double surface $S = \mathrm{sing}\,(X)$ is. This uses the above trick with lifting entire curves to the normalization $C^{(3)}$ of X.

• The irregularity $q(S) \geq g > 5$. This is based on the fact that for a curve C with general moduli, the Jacobian $J(C)$ is a simple abelian variety.

• S is hyperbolic iff it is algebraically hyperbolic that is, does not contain any rational or elliptic curve. This is based on the Bloch Conjecture.

• S is hyperbolic iff the triple curve $T \subseteq S$ of X is. Recall that in a general point of T, 3 smooth branches of X meet transversally. Actually T parameterizes the 3-secant lines of $C^{(3)} \subseteq \mathbb{P}^7$ parallel to the center of the projection $\mathbb{P}^7 \dashrightarrow \mathbb{P}^4$. The proof is based on Pirola's and Ciliberto-van der Geer's results on deformations of hyperelliptic and bielliptic curves on abelian varieties.

• Any irreducible component of the triple curve[1] T has genus ≥ 2. The proof is rather involved.

3. DEFORMATION METHOD

Let $X_0 = f_0^*(0)$, $X_\infty = f_\infty^*(0)$ be two hypersurfaces of the same degree d in $\mathbb{P}^n$, and let

$$\{X_t\}_{t\in\mathbb{P}^1} = \langle X_0, X_\infty\rangle, \qquad \text{where} \qquad X_t = (f_0 + tf_\infty)^*(0)\,,$$

be the pencil of hypersurfaces generated by X_0 and X_∞. For small enough $|\varepsilon| \neq 0$ we call X_ε a small (linear) deformation of X_0 in direction of X_∞.

DEFINITION We say that a (very) general small deformation of X_0 is hyperbolic if X_ε is for a (very) general X_∞ and for all sufficiently small $\varepsilon \neq 0$ (depending on X_∞).

Let us formulate the following

"Weak Kobayashi Conjecture" : *For every hypersurface $X \subseteq \mathbb{P}^n$ of degree $d \geq 2n-1$, a (very) general small deformation of X is Kobayashi hyperbolic.*

By Brody's Theorem, the proof of hyperbolicity of X reduces to a certain degeneration principle for entire curves in X. The Green-Griffiths' $79'$ proof of Bloch's Conjecture [7] provides a kind of such degeneration principle. It was shown by McQuillen [9] and, independently, by Demailly-El Goul [3] (according with this principle) that every entire curve $\varphi : \mathbb{C} \to X$ in a very general surface $X \subseteq \mathbb{P}^3$ of degree $d \geq 36$ ($d \geq 21$, respectively) satisfies a certain algebraic differential equation.

[1]Presumably T is irreducible, but we don't dispose a proof of this.

Consider again a pencil (X_t). Assuming that for a sequence $\varepsilon_n \to 0$ the hypersurfaces X_{ε_n} are not hyperbolic, one can find a sequence of Brody entire curves $\varphi_n : \mathbb{C} \to X_{\varepsilon_n}$ which converges to a (non-constant) Brody curve $\varphi : \mathbb{C} \to X_0$.

Suppose in addition that X_0 admits a rational map $\pi : X_0 \dashrightarrow Y_0$ to a hyperbolic variety Y_0 (to a curve Y_0 of genus ≥ 2 in case $\dim X_0 = 2$). Then necessarily $\pi \circ \varphi = \mathrm{cst}$, provided that the composition $\pi \circ \varphi$ is well defined. Anyhow the limiting Brody curve $\varphi : \mathbb{C} \to X_0$ degenerates. This degeneration however is not related to any specific property of the configuration $X_0 \cup X_\infty$, but of X_0 alone. Here is another degeneration principle which involves both X_0 and X_∞.

PROPOSITION 1 (Shiffman-Z′ ′05 [11]**, Z′ ′07** [16]**)** Consider a pencil of degree d hypersurfaces $X_\varepsilon \subseteq \mathbb{P}^{n+1}$ generated by $X_0 = X_0' \cup X_0''$ and X_∞. Let $D = X_0' \cap X_0''$. Then for any sequence of entire curves $\varphi_n : \mathbb{C} \to X_{\varepsilon_n}$ which converges to $\varphi : \mathbb{C} \to X_0'$ the following alternative holds:
• Either $\varphi(\mathbb{C}) \subseteq D$, or
• $\varphi(\mathbb{C}) \cap D \subseteq D \cap X_\infty$ and $d\varphi(t) \in T_P X_0' \cap T_P X_\infty \quad \forall P = \varphi(t) \in D \cap X_\infty$.

THEOREM 1 (Z′ ′07 [16]**)** *Let Y_0 be a Kobayashi hyperbolic hypersurface of degree d in $\mathbb{P}^n$ ($n \geq 2$), where $\mathbb{P}^n$ is realized as the hyperplane $H = \{z_{n+1} = 0\}$ in $\mathbb{P}^{n+1}$. Then a general small deformation $X_\varepsilon \subseteq \mathbb{P}^{n+1}$ of the double cone $2X_0$ over Y_0 is Kobayashi hyperbolic.*

The proof is based on Proposition 1 and on the following lemma.

LEMMA 1 *Let $\hat{Y} \subseteq \mathbb{P}^{n+1}$ be a cone over a projective variety $Y \subseteq \mathbb{P}^n$, and let $X' \subseteq \mathbb{P}^{n+1}$ be a general hypersurface of degree $e \geq 2\dim Y$. Then X' meets every generator l of $\hat{Y}$ in at least $k = e - 2\dim Y$ points transversally.*

Proof of Theorem 1. Suppose the contrary. Then we can find a sequence $\varepsilon_n \longrightarrow 0$ and a sequence of Brody curves $\varphi_n : \mathbb{C} \to X_{\varepsilon_n}$ such that $\varphi_n \longrightarrow \varphi$, where $\varphi : \mathbb{C} \to X_0$ is non-constant. We let $\pi : X_0 \dashrightarrow Y_0$ be the cone projection. Since Y_0 is assumed to be hyperbolic we have $\pi \circ \varphi = \mathrm{cst}$. In other words $\varphi(\mathbb{C}) \subseteq l$, where $l \cong \mathbb{P}^1$ is a generator of the cone X_0.

We note that $\bigtriangledown f_0^2|_{X_0} = 0$. If l and X_∞ meet transversally in a point $\varphi(t) \in l \cap X_\infty$ then $d\varphi(t) = 0$ by virtue of Proposition 1.

Since $Y_0 \subseteq \mathbb{P}^n$ is hyperbolic and $n \geq 2$ we have $d \geq n+2$. In particular

$$\deg X_\infty = 2d \geq 2n + 4 \geq 2\dim Y + 5\,.$$

By Lemma 1, l and X_∞ meet transversally in at least 5 points. Hence the nonconstant meromorphic function $\varphi : \mathbb{C} \to l \cong \mathbb{P}^1$ possesses at least 5 multiple values. Since the defect of a multiple value is $\geq 1/2$, this contradicts the Defect Relation. $\square$

REMARK Given a hyperbolic hypersurface $Y \subseteq \mathbb{P}^n$ of degree d, Theorem 1 provides a hyperbolic hypersurface $X \subseteq \mathbb{P}^{n+1}$ of degree $2d$. Iterating the construction yields hyperbolic hypersurfaces in $\mathbb{P}^n$ $\forall n \geq 3$ of degree that grows exponentially with n.

EXAMPLE (Z′ ′07 [16]**)** Let $C \subseteq \mathbb{P}^2$ be a hyperbolic curve of degree $d \geq 4$, and let $X_0 \subseteq \mathbb{P}^3$ be a cone over C. Then a general small deformation of the double cone $2X_0$ is a Kobayashi hyperbolic surface in $\mathbb{P}^3$ of even degree $2d \geq 8$.

The following example combines the projection and deformation methods.

EXAMPLE (Shiffman-Z′ ′03 [12]**)** There is a singular octic $X_0 \subseteq \mathbb{P}^3$ whose normalization is a simple abelian surface. Moreover, a general small deformation of X_0 is Kobayashi hyperbolic.

EXAMPLE (Shiffman-Z′ ′05 [11]**)** Let $X_0 = X_0' \cup X_0''$ be the union of two cones in general position in $\mathbb{P}^3$ over smooth plane quartics C', $C'' \subseteq \mathbb{P}^2$, respectively. Then a general small deformation of X_0 is Kobayashi hyperbolic.

Sketch of the proof. Suppose that for a sequence $\varepsilon_n \to 0$, X_{ε_n} is not hyperbolic. Then we can find a sequence of Brody curves $\varphi_n : \mathbb{C} \to X_{\varepsilon_n}$ which converges to a Brody curve $\varphi : \mathbb{C} \to X_0$. We may assume that $\varphi(\mathbb{C}) \subseteq X_0'$.

Since C' has genus 3, $\pi' \circ \varphi : \mathbb{C} \to C'$ is constant, where $\pi' : X_0' \dashrightarrow C'$ is the cone projection. Thus $\varphi(\mathbb{C}) \subseteq l$, where l is a generator of the cone X_0'.

By Proposition 1, $\varphi(\mathbb{C})$ meets the double curve $D = X_0' \cap X_0''$ of X_0 only in points of $D \cap X_\infty$. The projection $\pi' : D \to C'$ has degree $d'' = 4$ and simple ramifications. Hence every fiber of $\pi'|D$ contains at least 3 points. A general octic X_∞ does not meet the ramification fibers of $\pi' : D \to C'$ and crosses D passing through just one point of the corresponding fiber of $\pi'|D$. Therefore $D \backslash X_\infty$ contains at least 3 points of l. According to the Little Picard Theorem, $\varphi : \mathbb{C} \to l \backslash (D \backslash X_\infty)$ is constant, a contradiction.

The Degeneration Principle of Proposition 1 can be combined with the following one.

PROPOSITION 2 (Z′ 07′ [16]**)** *Let $(X_t)_{t\in\mathbb{P}^1}$ be a pencil of hypersurfaces in $\mathbb{P}^{n+1}$ generated by two hypersurfaces X_0 and X_∞ of the same degree $d \geq 5$, where $X_0 = kQ$ with $k \geq 2$ for some hypersurface $Q \subseteq \mathbb{P}^{n+1}$, and $X_\infty = \bigcup_{i=1}^d H_{a_i}$, $a_1, \ldots, a_d \in \mathbb{P}^1$, is a union of d distinct hyperplanes from a pencil $(H_a)_{a\in\mathbb{P}^1}$. If a sequence of Brody curves $\varphi_n : \mathbb{C} \to X_{\varepsilon_n}$, where $\varepsilon_n \to 0$, converges to a Brody curve $\varphi : \mathbb{C} \to X_0$, then $\varphi(\mathbb{C}) \subseteq X_0 \cap H_a$ for some $a \in \mathbb{P}^1$.*

EXAMPLES Given a pencil of planes (H_a) in $\mathbb{P}^3$, using Proposition 2 one can deform
- $X_0 = 5Q$, where $Q \subseteq \mathbb{P}^3$ is a plane,
- a triple quadric $X_0 = 3Q \subseteq \mathbb{P}^3$, or
- a double cubic, quartic, etc. $X_0 = 2Q \subseteq \mathbb{P}^3$

to an irreducible surface $X_\varepsilon \in \langle X_0, X_\infty \rangle$ of the same degree d, where as before $X_\infty = \bigcup_{i=1}^d H_{a_i}$, so that every limiting Brody curve $\varphi : \mathbb{C} \to X_0$ is contained in a section $X_0 \cap H_a$ for some $a \in \mathbb{P}^1$.

The famous Bogomolov-Green-Griffiths-Lang Conjecture on strong algebraic degeneracy (see e.g., [1, 7]) suggests that every surface S of general type possesses only finite number of rational and elliptic curves and, moreover, the image of any nonconstant entire curve $\varphi : \mathbb{C} \to S$ is contained in one of them. In particular, this should hold for any smooth surface $S \subseteq \mathbb{P}^3$ of degree ≥ 5, which fits the Kobayashi Conjecture. Indeed, by Clemens-Xu-Voisin's Theorem, a general smooth surface $S \subseteq \mathbb{P}^3$ of degree ≥ 5 does not contain rational or elliptic curves, hence should be hyperbolic. Anyhow, the deformation method leads to the following result, which is an immediate consequence of Proposition 2.

COROLLARY *Let $S \subseteq \mathbb{P}^3$ be a surface and $Z \subset S$ be a curve such that the image of any nonconstant entire curve $\varphi : \mathbb{C} \to S$ is contained in Z* [2]*. Let X_∞ be the union of $d = 2 \deg S$ planes from a general pencil of planes in $\mathbb{P}^3$. Then any small enough linear deformation X_ε of $X_0 = 2S$ in direction of X_∞ is hyperbolic.*

Along the same lines, Proposition 2 applies in the following setting.

EXAMPLE Let us take for X_0 a double cone in $\mathbb{P}^3$ over a plane hyperbolic curve of degree $d \geq 4$, and for X_∞ a union of $2d$ distinct planes from a general pencil (H_a). Then small deformations X_ε of X_0 in direction of X_∞ provide examples of hyperbolic surfaces of any even degree $2d \geq 8$. For $d = 4$ the latter surfaces can be given by equation (1) in suitable coordinates. Hence these are actually the Duval-Fujimoto examples [5, 6].

A nice construction due to J. Duval '04 [4] of a hyperbolic sextic $X_\varepsilon \subseteq \mathbb{P}^3$ uses the deformation method iteratively in 5 steps, so that $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_5)$ has 5 subsequently small enough components. Hence X_ε vary within a 5-dimensional linear system; however the deformation of X_0 to X_ε neither is linear nor very generic. It was suggested in [12] that the union of 6 general planes in $\mathbb{P}^3$ admits a general small linear deformation to an irreducible hyperbolic sextic surface.

Let us finally turn to the Kobayashi problem on hyperbolicity of complements of general hypersurfaces. By virtue of Kiernan-Kobayashi-M. Green's version of Borel's Lemma, the complement $\mathbb{P}^n \setminus L$ of the union $L = \bigcup_{i=1}^{2n+1} L_i$ of $2n+1$ hyperplanes in $\mathbb{P}^n$ in general position is Kobayashi hyperbolic. In particular, this applies to the union l of 5 lines in $\mathbb{P}^2$ in general position. Moreover [17] l can be deformed to a smooth quintic curve with hyperbolic complement via a small deformation. This deformation proceeds in 5 steps and neither is linear nor very generic. So the following question arises.

Question. Let L (M) stands for the union of $2n+1$ $(2n-1$, respectively) hyperplanes in $\mathbb{P}^n$ in general position. Is the complement of a general small linear deformation of L Kobayashi hyperbolic? Is a general small linear deformation of M Kobayashi hyperbolic? In particular, does the union of 5 lines in $\mathbb{P}^2$ (of 5 planes in $\mathbb{P}^3$) in general position admit a general small linear deformation to an irreducible quintic curve with hyperbolic complement (to a hyperbolic quintic surface, respectively)?

[2]The latter holds, for instance, if S is hyperbolic modulo Z.

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